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An Introduction to Delay Differential Equations with Applications to the Life Sciences

-- Monograph --

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Preface

This book is intended to be an introduction to delay differential equations for upper-level undergraduates or beginning graduate mathematics students who have a reasonable background in ordinary differential equations and who would like to get to the applications quickly. I used a preliminary version of this manuscript in teaching such a course at Arizona State University over the past two years. Existing texts on the subject by Diekmann et al. [26] and by Hale and Lunel [41], while excellent on the theory, are heavy on functional analytic background and light on applications. In my experience, most graduate students do not have the requisite background to read such texts profitably. A more applications oriented text by Kuang [48] is, unfortunately, out of print.

Both theory and applications of delay differential equations require a bit more mathematical maturity than its ordinary differential equations counterparts. Primarily, this is because the theory of complex variables plays such a large role in analyzing the characteristic equations that arise on linearizing around equilibria. Ideal prerequisites for this book include a second course in ordinary differential equations such as in the text [78, 10], some familiarity with complex variables, and some elementary analysis. Results from the calculus of several variables are routinely used, especially, the implicit function theorem.

This book focuses on the key tools necessary to understand the applications literature involving delay equations and to construct and analyze mathematical models involving delay differential equations. It begins with a survey of mathematical models involving delay equations. These are primarily from the biological literature, in keeping with my own prejudices, and due to the relative frequency of delay models in that literature relative to others. This is followed by a “warm-up” chapter on the simplest possible delay equation \( u'(t) = -\alpha u(t - r) \). This simple example illustrates many of the complexities that arise with delays and has the advantage that results may be easily and explicitly worked out. Its main message is that delays naturally induce oscillations. Standard existence and uniqueness results are taken up in Chapter 3. The method of steps is introduced and exploited for discrete delay equations. For the reader interested mainly in applications, this may suffice. A more general approach follows but no fixed point theorems are used: the method of successive
approximations works fine. A key notation is introduced here, one that takes a bit of
getting used to, namely the state variable $x$, which appears throughout the remain-
der of the book. In addition to continuous dependence of solutions on initial data,
continuation of solutions, positivity, and comparison of solutions are also discussed
because many applications come from biology where positivity restrictions are in-
herent to the models. Linear equations are taken up next with the primary aim being
stability. In applications, linear delay equations arise through linearization of a non-
linear equation about an equilibria so the focus is on linear stability analysis and the
characteristic equation the roots for which determine stability. Proof of the validity
of linearized stability would require too much additional mathematics and therefore
it is not given.

The following chapter is an introduction to abstract dynamical systems theory,
using ordinary differential equations, discrete-time difference equations, and now
delay differential equations as examples. It is shown that a delay differential equa-
tion induces a semidynamical system on the space of continuous functions on the
delay interval. The focus then turns to omega limit sets, the usual results familiar
from ODEs continue to hold but with some nuances due to the infinite-dimensional
state space. Applications to the delayed logistic equation and the delayed chemostat
model are treated. The LaSalle invariance principle is established and an applica-
tion is given. Next, the Hopf bifurcation theorem, critical for applications, is treated.
A simple canonical example is considered where the bifurcation can be explicitly
computed. Following this, the Hopf bifurcation theorem is stated without proof. It
is applied to the standard delayed negative feedback system
$$x'(t) = -f(x(t-1))$$
where $xf(x) > 0$. In this case, a formal expansion for the periodic solution in terms
of a small parameter (this is fully justified in an appendix) is given. Applications
to various second-order delay equations are then considered, one of which is stabi-
лизирующая установившихся значений дифференциальной системы с запаздыванием;
another is a model of a gene regulatory network. Finally, the beautiful Poincaré–Bendixson
theory for monotone cyclic feedback systems, obtained recently by Mallet-Paret and
Sell, is stated.

The following brief chapter is an introduction to equations with infinite delay and
to the linear chain trick by which certain special kinds of infinite delays can lead to
ordinary differential equations. These arise often in the modeling literature so an
example is discussed in some detail. The final chapter focuses on a model of virus
predation on a bacterial host in the setting of a chemostat where the bacteria subsist
on a supplied nutrient. The delay corresponds to the latent period following virus
infection during which new virus particles are manufactured within the cell. Most
of the theoretical results of previous chapters are used in the analysis of this system
of delay equations.

Two brief appendices should help those readers needing additional background
on complex variables and analytic functions including the very useful Rouché’s the-
orem, and implicit function theorems. The Ascoli–Arzelà theorem is stated and dis-
cussed and the useful fluctuation method is described. A second appendix is devoted
to a rigorous proof of Hopf bifurcation for the delayed negative feedback systems.
The impatient reader could skim the applications in Chapter 1, jump over Chapter 2, and start with Chapter 3. A note on notation: we use $\mathbb{R}$ for the set of real numbers, $\mathbb{C}$ for the set of complex numbers, and $f'$ denotes the derivative of a function $f$.

I would like to acknowledge the influence of Yang Kuang, a collaborator on much of the author’s own work in delay differential equations, on this work and to thank him for providing several figures used in the book. Several students, colleagues, and anonymous reviewers read portions of the manuscript and provided valuable feedback. Among these, the author would like to thank Patrick de Leenheer, Thanate Dhirasakdanon, Zhuo Han, and Harlan Stch. Most of what I know about delay differential equations, I learned from Jack Hale, a giant in the field.

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Contents

1 Introduction ......................................................... 1
  1.1 Examples of Delay Differential Equations .................. 1
  1.2 Some Terminology .......................................... 9
  1.3 Solving Delay Equations Using a Computer .............. 11

2 Delayed Negative Feedback: A Warm-Up ...................... 13
  2.1 Preliminaries .............................................. 13
  2.2 The Simplest Delay Equation .............................. 16
  2.3 Oscillation of Solutions .................................. 20
  2.4 Solutions Backward in Time ................................ 22

3 Existence of Solutions .......................................... 25
  3.1 The Method of Steps for Discrete Delay Equations ....... 25
  3.2 Positivity of Solutions .................................... 27
  3.3 A More General Existence Result ......................... 29
  3.4 Continuation of Solutions .................................. 36
  3.5 Remarks on Backward Continuation ....................... 37
  3.6 Stability Definitions ...................................... 38

4 Linear Systems and Linearization .............................. 41
  4.1 Autonomous Linear Systems ................................ 41
  4.2 Laplace Transform and Variation of Constants Formula ... 43
  4.3 The Characteristic Equation ............................... 45
  4.4 Small Delays Are Harmless ................................ 48
  4.5 The Scalar Equation $x'(t) = Ax(t) + Bx(t - r)$ ....... 49
  4.6 Principle of Linearized Stability ......................... 54
  4.7 Absolute Stability ......................................... 56

5 Semidynamical Systems and Delay Equations ................. 61
  5.1 The Dynamical Systems Viewpoint ......................... 61
  5.2 Semiflows and Omega Limit Sets .......................... 64
5.3 SemiDynamical Systems Induced by Delay Equations ............... 65
5.4 Monotone Dynamics ........................................... 70
5.5 Delayed Logistic Equation ........................................ 73
5.6 Delayed Microbial Growth Model .................................... 76
5.7 Liapunov Functions .................................................. 78
  5.7.1 Logistic Equation with Instantaneous and Delayed Density Dependence ................................................. 80

6 Hopf Bifurcation ...................................................... 87
  6.1 A Canonical Example ............................................... 87
  6.2 Hopf Bifurcation Theorem .......................................... 89
  6.3 Delayed Negative Feedback ......................................... 91
    6.3.1 Computation of the Hopf Bifurcation ......................... 92
    6.3.2 Series Expansion of Hopf Solution ......................... 94
    6.3.3 The Logistic Equation ........................................ 97
  6.4 A Second-Order Delayed Feedback System ......................... 99
    6.4.1 Delayed Feedback Dominates Instantaneous Feedback ....... 101
    6.4.2 Instantaneous Feedback Dominates Delayed Feedback ...... 104
    6.4.3 Stabilizing the Straight-Up Steady State of the Pendulum .... 106
  6.5 Gene Regulation by End-Product Repression ....................... 111
  6.6 A Poincaré-Bendixon Theorem for Delay Equations ................. 115

7 Distributed Delay Equations and the Linear Chain Trick ............. 119
  7.1 Infinite Delays of Gamma Type .................................... 119
    7.1.1 Characteristic Equation and Stability ...................... 120
    7.1.2 The Linear Chain Trick ..................................... 123
  7.2 A Model of HIV Transmission ...................................... 126
  7.3 An ODE Approximation to a Delay Equation .......................... 129

8 Phage and Bacteria in a Chemostat .................................. 131
  8.1 Model .................................................................... 131
  8.2 Positivity and Boundedness of Solutions ........................... 133
  8.3 Basic Reproductive Number for Phage .............................. 134
  8.4 Persistence of Host and Phage Extinction ......................... 135
  8.5 The Coexistence Equilibrium ...................................... 137
  8.6 Another Formulation of the Model ................................ 141

A Results from Real and Complex Analysis .............................. 149
  A.1 Analytic Functions .................................................. 149
  A.2 Implicit Function Theorem for Complex Variables ............... 151
  A.3 Rouché’s Theorem .................................................. 152
  A.4 Ascoli–Arzelà Theorem ............................................ 153
  A.5 Fluctuation Lemma .................................................. 154
  A.6 General Implicit Function Theorem ................................. 155
  A.7 Gronwall’s Inequality .............................................. 155
B  Hopf Bifurcation for Delayed Negative Feedback 157
   B.1  Basic Setup and Preliminaries 157
   B.2  The Solution 160
      B.2.1  Solve for q 161
      B.2.2  Solve for μ and δ 163
References 167
Index 171