Winning@Wimbledon

Milan Pahor

Abstract The usual scoring system used in tennis games is investigated mathematically using binomial probability theory. It is shown that this scoring system amplifies the superiority of the more skilled player. For example a player with a 55% chance of winning any individual point enjoys a more than 95% chance of winning the match. A new tennis scoring system, in which deuce is eliminated and the game is won by the first player to gain a single point past 40-40, is also considered in this way. This new scoring system would lessen the amplification of skill level. For example, in this new scoring system, a player with a 55% chance of winning any individual point now has less than 90% chance of winning the match.

The scoring system in the game of tennis is a remarkably complex and elegant structure. To secure a game a player must win at least 4 points (curiously referred to as 15-30-40-game). In the event that both players reach three points (that is 40-40 or “deuce”) a player must then win two consecutive points in order to win the game, with the score returning to deuce if a player wins only one of the next two points. (The term deuce is a corruption of the French deux for two). A consequence of this unique mechanism is that a single game of tennis could theoretically never finish! Games are accumulated to sets and at Wimbledon the men’s final is decided over five sets with the first four being tiebreaker sets and the fifth advantage. (The difference between tie-breaker and advantage sets will be discussed later)

The function of any scoring system is of course to keep track of who is ahead and to decide upon the eventual winner. A much more subtle purpose is to amplify the superiority of the better player and thus to minimise the probability of victory by chance as opposed to skill. At first glance it seems reasonable to assume that a player with a 60% chance of winning any individual point will also have the same chance of winning a five set match, but this is far from the truth! In this article we will use simple binomial probability theory to show that a player at Wimbledon with a 60% chance of winning any individual point enjoys a massive 99.96% chance of winning the match! We begin with a quick revision of binomial coefficients and binomial probability. If you are familiar with this material you may wish to go directly to section 3) Winning at Wimbledon.

1. Binomial Coefficients

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Consider the simple question “In how many different ways can three people A, B and C line up for a bus?” Straightforward listing ABC, ACB, BAC, BCA, CAB, CBA yields an answer of 6. A much more versatile argument is that there are three people to fill the first place, two for the second (since one person has been used up) and one for the last, implying an answer of 3.2.1 = 6 as before. We refer to the quantity 3.2.1 as 3 factorial and denote it by 3!.

Definition If \( n \) is a positive integer then \( n! = n(n−1)(n−2)(n−3) \cdots 1 \)

We can interpret \( n! \) as the total number of ways of permuting \( n \) objects (with \( 0! = 1 \) by convention). Your calculator will almost certainly have an \( n! \) button. The advantage of factorial notation is that large questions such as “In how many different ways can 10 people line up for a bus?” are tractable. The answer is simply 10! = 10 · 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1 = 3,628,800. Imagine having to list the options! Consider now the slightly different question “In how many ways can a team of three people be chosen from a group A, B, C, D, E of five people?” Simple listing

\( ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE \)

leads to an answer of 10. Note that we do not include both \( ABC \) and \( ACB \) since we are now interested in combinations not permutations. That is team \( ABC \) is the same as team \( ACB \). Once again the above listing approach is clumsy. A more streamlined attack is to reintroduce order and argue that the first position can be filled in 5 different ways, the second in 4 different ways and the final place in 3 different ways giving us 5.4.3 = 60 ordered triples. (Note that we can express 5.4.3 as \( \frac{5!}{2!} = \frac{5.4.3.2.1}{2.1} \)). But many of these arrangements are equivalent. For example team \( ABC \approx ACB \approx BAC \approx BCA \approx CAB \approx CBA \). Each ordered team will be in an equivalence class of 6, so we need to factor out 60 by 6(=3!) and as before we get 10 \( \left( \frac{60}{6} \right) \) different teams.

We refer to the quantity 10 = \( \frac{5!}{3!2!} \) by \( ^5C_3 \) read as 5 choose 3. In general if \( n \) and \( r \) are non-negative integers with \( r \leq n \) then the Binomial coefficient \( ^nC_r = \frac{n!}{r!(n−r)!} \) is the number of different ways of choosing a team of \( r \) from a group of \( n \). An alternative notation for \( ^nC_r \) is \( \binom{n}{r} \). Thus \( ^7C_4 = \binom{7}{4} = \frac{7!}{4!3!} = 35 \) is the number of different ways of choosing 4 from 7. Lurking somewhere on your calculator you will find an \( ^nC_r \) button.

Exercise: Prove algebraically that \( ^nC_r = ^nC_{n−r} \) and explain this result in terms of team selection.

2. Binomial Probability

Suppose that the probability of a basketballer hitting a basket is 0.6 and consider the question “What is the probability that she hits exactly 3 baskets in 8 attempts?” Let us first answer a simpler question “What is the probability that her first 3 shots are hits and the next 5 are misses?” This is easy:

\[
P(\text{hhmmmmmm}) = 0.6 \times 0.6 \times 0.6 \times 0.4 \times 0.4 \times 0.4 \times 0.4 = (0.6)^3 (0.4)^5
\]
The difference between the two questions is that in the second the position of the three hits is specified whereas in the original question the three hits could occur in any of the eight shots. In how many ways can we choose the position of the 3 hits out of 8? Well it’s \(8C_3\) of course! So the answer to the original problem is \(8C_3(6)^3(4)^5\). In general if an experiment has a probability of \(p\) of success and \(q = 1 - p\) of failure then the binomial probability of exactly \(r\) successes in \(n\) trials is given by \(nC_r(p)^r(q)^{n-r}\).

3. Winning at Wimbledon

We now have the required tools to analyse the scoring system in tennis. Assuming a player has a probability of \(p\) of winning any individual point we seek to determine his probability \(w\) of winning a standard Wimbledon match as a function \(w = f(p)\) of \(p\). A closed form for \(w = f(p)\) will be established and properties of the function will be investigated. Deuce plays a central and intriguing role and we will explore how the elimination of this feature (as is currently being advocated) affects the distribution of probabilities. The current structure will be directly compared with that before tiebreakers and also with scoring systems planned for the future.

**Probability of winning a game:**

Let us assume that a player has a probability of \(p\) of winning any point in the match and thus \(q = 1 - p\) of losing a point. A player can win a game from 40-0, 40-15, 40-30 or from deuce.

**40-0:** To win from 40-0 down a player needs to win 4 consecutive points: \((p)^4\)

**40-15:** To win from 40-15 a player needs to win 3 out of the first 4 points and then win the next point: \(4C_3(p)^3(q) \times (p) = 4C_3(p)^4(q) = 4(p)^4(4)\).

The term \(4C_3(p)^3(q)\) is simply the binomial probability of 3 successes out of 4. It may be tempting to argue that to win from 40-15 a player simply needs to win 4 points out of 5 with resulting probability \(5C_4(p)^4(q)\). This is incorrect since it includes the event \(wwwwl\), however if the first four points are won there is no fifth point at all! In all the subsequent analysis keep in mind that to win a game you must have won the last point of that game, to win a set you must have won the last game of that set and to win a match you must have won the last set of that match.

**40-30:** To win from 40-30 a player needs to win 3 out of the first 5 points and then win the next point: \(5C_3(p)^3(q)^2 \times p = 5C_3(p)^4(q)^2 = 10(p)^4(4)^2\).

**Deuce:** This is interesting! The probability of getting to deuce is \(6C_3(p)^3(q)^3\) (3 points out of 6). Once deuce is reached the game can be secured by winning two successive points, or returning to deuce and then winning two successive points or returning to deuce twice and then winning two in a row etc. The probability of reaching deuce again is \(2pq\) (win-loss or loss-win). Thus the probability of winning a deuce game is

\[
6C_3(p)^3(q)^3\{p^2 + 2(pq)p^2 + (2pq)^2p^2 + \cdots (2pq)^n p^2 + \cdots\}
\]

\[
= 6C_3(p)^5(q)^3\{1 + (2pq) + (2pq)^2 + \cdots (2pq)^n + \cdots\}
\]

and since the quantity in the brackets is a converging (can you prove \(-1 < 2pq < 1\)?)
geometric progression with $a = 1, r = 2pq$ and $S_\infty = \frac{a}{1-r}$ we have a probability of

$$6C_3(p)^5(q)^3 \left\{ \frac{1}{1-2pq} \right\} = 20(p)^5(q)^3 \left\{ \frac{1}{1-2pq} \right\}$$

If you are unfamiliar with the theory of sequences and series we may also proceed as follows:

Let $x$ be the probability of winning a game assuming that we are standing at deuce. Then $x = p^2 + 2pq x$. That is to win from deuce we must either win the next two points or return to deuce and then win from deuce. Thus $x - 2pqx = p^2$ implying that $x(1-2pq) = p^2$ and hence $x = \frac{p^2}{1-2pq}$ yielding a probability of

$$6C_3(p)^3(q)^3 \left\{ \frac{p^2}{1-2pq} \right\} = 20(p)^3(q)^3 \left\{ \frac{1}{1-2pq} \right\}$$

as before.

Thus the probability of winning a game is $g = (p)^4 + 4(p)^4(q) + 10(p)^4(q)^2 + 20(p)^5(q)^3 \left\{ \frac{1}{1-2pq} \right\}$, with $h = 1 - g$ being that of losing a game.

**Probability of winning an advantage set: $s$**

To win an advantage set a player needs to reach 6 games with a margin of 2. The respective probabilities are:

- **6-0**: $g^6$
- **6-1**: $6C_5(g)^5(h) \times g = 6(g)^6(h)$
- **6-2**: $7C_5(g)^5(h)^2 \times g = 21(g)^6(h)^2$
- **6-3**: $8C_5(g)^5(h)^3 \times g = 56(g)^6(h)^3$
- **6-4**: $9C_5(g)^5(h)^4 \times g = 126(g)^6(h)^4$

Just as with deuce an advantage set could theoretically never terminate. A similar argument to that presented above yields

$$\{7-5, 8-6, 9-7 \cdots \} : 10C_5(g)^7(h) \{ g^2 + (2gh)g^2 + (2gh)^2g^2 + \cdots (2gh)^ng^2 + \cdots \}$$

$$= 252(g)^7(h)^5 \left\{ \frac{1}{1-2gh} \right\}.$$ 

Thus the probability of winning an advantage set is

$$s = (g)^6 + 6(g)^6(h) + 21(g)^6(h)^2 + 56(g)^6(h)^3 + 126(g)^6(h)^4 + 252(g)^7(h)^5 \left\{ \frac{1}{1-2gh} \right\}$$

where $g$ and $h$ are the probabilities of winning and losing a game defined above.

We let $t = 1 - s$ be the probability of losing an advantage set.

**Probability of winning a tie-breaker set: $u$**
A perceived problem with advantage sets is that they could theoretically go forever. At the 1969 championships Pancho Gonzales beat Charlie Pasarell 22-24, 1-6, 16-14, 6-3, 11-9 in five hours and 12 minutes. In the early 70’s tiebreakers were introduced at 6 games each in order to terminate sets quickly. A tiebreaker is simply a sequence of \textbf{individual points} with the winner of the set being the first to reach 7 points with a margin of 2. Thus tiebreakers can be won \{7-0, 7-1, 7-2, 7-3, 7-4, 7-5, 8-6, 9-7, 10-8 \cdots \}.

We leave as an exercise that the probability $u$ of winning a tiebreaker set is

$$u = g^6 + 6(g)^6(h) + 21(g)^6(h)^2 + 56(g)^6(h)^3 + 126(g)^6(h)^4 + 252(g)^7(h)^5$$

$$+ 504g^6h^6 \left\{ p^7 + 7p^7q + 28p^7q^2 + 84p^7q^3 + 210p^7q^4 + 462p^7q^5 + \frac{924p^8q^6}{1 - 2pq} \right\}$$

with $v = 1 - u$ being the probability of losing a tiebreaker set. Observe that $u$ is a function of both $g$ and $p$ since both games and individual points play a role in the outcome of a tiebreaker set.

**Probability of winning a match: $w$**

The men’s final at Wimbledon is played over 5 sets with the first 4 being tiebreaker sets and the fifth set being advantage. The first player to secure 3 sets is declared the winner. The probabilities are:

3-0: \(u^3\)
3-1: \(3C_2(u)^2(v) \times u = 3(u)^3(v)\)
3-2: \(4C_2(u)^2(v)^2 \times s = 6(u)^2(v)^2s\)

Hence the probability of winning at Wimbledon is $w = u^3 + 3(u)^3(v) + 6(u)^2(v)^2s$ where $u, v$ and $s$ are as defined above.

What is the probability $w$ of winning the match as a function $w = f(p)$? Well simple
calculation yields:

\[
\begin{aligned}
    w &= ((p^4 + 4 \cdot p^4 \cdot (1 - p) + 10 \cdot p^4 \cdot (1 - p)^2 + 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^6 + 6 \cdot \\
    &\quad (p^4 + 4 \cdot p^4 \cdot (1 - p) + 10 \cdot p^4 \cdot (1 - p)^2 + 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^6 \times \\
    &\quad (1 - p^4 - 4 \cdot p^4 \cdot (1 - p) - 10 \cdot p^4 \cdot (1 - p)^2 - 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^2 + \\
    &\quad 21 \cdot (p^4 + 4 \cdot p^4 \cdot (1 - p) + 10 \cdot p^4 \cdot (1 - p)^2 + 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^6 \times \\
    &\quad (1 - p^4 - 4 \cdot p^4 \cdot (1 - p) - 10 \cdot p^4 \cdot (1 - p)^2 - 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^2 + \\
    &\quad 56 \cdot (p^4 + 4 \cdot p^4 \cdot (1 - p) + 10 \cdot p^4 \cdot (1 - p)^2 + 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^6 \times \\
    &\quad (1 - p^4 - 4 \cdot p^4 \cdot (1 - p) - 10 \cdot p^4 \cdot (1 - p)^2 - 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^2 + \\
    &\quad 126 \cdot (p^4 + 4 \cdot p^4 \cdot (1 - p) + 10 \cdot p^4 \cdot (1 - p)^2 + 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^6 \times \\
    &\quad (1 - p^4 - 4 \cdot p^4 \cdot (1 - p) - 10 \cdot p^4 \cdot (1 - p)^2 - 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^2 + \\
    &\quad 210 \cdot (p^4 + 4 \cdot p^4 \cdot (1 - p) + 10 \cdot p^4 \cdot (1 - p)^2 + 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^6 \times \\
    &\quad (1 - p^4 - 4 \cdot p^4 \cdot (1 - p) - 10 \cdot p^4 \cdot (1 - p)^2 - 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^2 + \\
    &\quad 252 \cdot (p^4 + 4 \cdot p^4 \cdot (1 - p) + 10 \cdot p^4 \cdot (1 - p)^2 + 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^6 \times \\
    &\quad (1 - p^4 - 4 \cdot p^4 \cdot (1 - p) - 10 \cdot p^4 \cdot (1 - p)^2 - 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^2 + \\
    &\quad 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^7 \times (1 - p^4 - 4 \cdot p^4 \cdot (1 - p) - 10 \cdot p^4 \cdot (1 - p)^2 - 20 \cdot p^5 \cdot (1 - p)^3/(1 - 2 \cdot p \cdot (1 - p)))^2
\end{aligned}
\]
\[-20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))\] 
\[+ 504 \cdot (p^4 + 4 \cdot p^4 \cdot (1-p) + 10 \cdot p^4 \cdot (1-p)^2 + 20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))^5 \cdot (1-p)^2 + 20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))\] 
\[+ 6 \cdot (p^4 + 4 \cdot p^4 \cdot (1-p) + 10 \cdot p^4 \cdot (1-p)^2 + 20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))^5 \cdot (1-p)^2 + 20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))^6 \cdot (1-p)^2 + 20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))^6 \cdot (1-p)^2 + 20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))^6 \cdot (1-p)^2 + 20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))^6 \cdot (1-p)^2 + 20 \cdot p^5 \cdot (1-p)^3/(1-2 \cdot p \cdot (1-p))^6 \]
The above function $w = f(p)$ is somewhat mind-boggling but it is just a function with properties and a graph! A computer analysis of $w = f(p)$ will clarify the picture.

Consider some properties we would expect $w = f(p)$ to have. Firstly $f(0.5)$ should certainly be 0.5. That is if you have a 50-50 chance of winning a point you should also have an even chance of winning the match. There should also be some form of skew symmetry about $(p = 0.5, w = 0.5)$ since the probability of winning the match at $p = 0.6$ must be the same as the probability of losing the match at $p = 0.4$. Consider the following computer generated table and graph of $p$ vs $w$:

\[
p^7 * (1 - p)^4 + 462 * p^7 * (1 - p)^5 + 924 * p^8 * (1 - p)^6 / (1 - 2 * p * (1 - p)))^2 * \\
(p^4 + 4 * p^4 * (1 - p) + 10 * p^4 * (1 - p)^2 + 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^6 \\
+ 6 * (p^4 + 4 * p^4 * (1 - p) + 10 * p^4 * (1 - p)^2 + 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^6 \\
(1 - p^4 - 4 * p^4 * (1 - p) - 10 * p^4 * (1 - p)^2 - 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p))) \\
+ 21 * (p^4 + 4 * p^4 * (1 - p) + 10 * p^4 * (1 - p)^2 + 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^6 \\
(1 - p^4 - 4 * p^4 * (1 - p) - 10 * p^4 * (1 - p)^2 - 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^2 + 56 * \\
(p^4 + 4 * p^4 * (1 - p) + 10 * p^4 * (1 - p)^2 + 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^6 \\
(1 - p^4 - 4 * p^4 * (1 - p) - 10 * p^4 * (1 - p)^2 - 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^3 \\
+ 126 * (p^4 + 4 * p^4 * (1 - p) + 10 * p^4 * (1 - p)^2 + 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^6 \\
(1 - p^4 - 4 * p^4 * (1 - p) - 10 * p^4 * (1 - p)^2 - 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^4 \\
+ 252 * (p^4 + 4 * p^4 * (1 - p) + 10 * p^4 * (1 - p)^2 + 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^7 \\
(1 - p^4 - 4 * p^4 * (1 - p) - 10 * p^4 * (1 - p)^2 - 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p)))^5 \\
(1 - 2 * (p^4 + 4 * p^4 * (1 - p) + 10 * p^4 * (1 - p)^2 + 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p))) \\
(1 - p^4 - 4 * p^4 * (1 - p) - 10 * p^4 * (1 - p)^2 - 20 * p^5 * (1 - p)^3 / (1 - 2 * p * (1 - p))))
\]
It is clear from the table that a 60% chance of winning an individual point certainly does not translate into a 60% chance of winning the match! In fact it results in a whopping 99.96% chance of victory. Observe also that a simple 1% increase in performance on points (50% to 51%) results in a 13% increase (50% to 63.3%) in matches won! What this means is that the scoring system does a wonderful job in ensuring that the superior player wins the match.

The graph of $w = f(p)$ is fascinating. It is certainly clear that $f(0.5) = 0.5$ and that there is skew symmetry at $p = 0.5$. Note also that the graph is steep at $p = 0.5$, a feature that any good scoring system should have! That is small changes $\Delta p$ around $p = 0.5$ should result in large changes of the probability of winning. The gradient at $p = 0.5$ is approximately 13.5. Consider the following graph of the derivative $\frac{dw}{dp}$.

![Graph showing steep gradient at $p = 0.5$](image)

Note the symmetry and how all the change occurs in a tight band around $p = 0.5$. It can be reasonably argued that the purpose of a scoring system is to make the spike as tall and as thin as possible!

In the good old days before the needs of television and advertising dictated the structure of sport, Wimbledon was played over 5 advantage sets with no tiebreakers.... no one cared if the match went for 6 hours! Let us call this era $w_{\text{past}}$. There is currently considerable pressure to shorten matches even further to three tiebreaker sets with no deuce at all! This means that once players reached 40-40 a single further point would decide the game. Let us call this world $w_{\text{future}}$. We close this article by comparing the data and rates of change for $w_{\text{past}}$ and $w_{\text{future}}$ with the above current state $w_{\text{now}}$.

Observe how the recent changes in the game favour the weaker player! A player with only a 46% chance of winning a point has their probability of victory moving from 8.37% to 8.86% to 17.15%. There is of course a corresponding erosion of the superior (54%) player’s chance of victory from 91.63% to 91.14% to 82.85%.
The derivative of \( w = f(p) \) at \( p = 0.5 \) (our measure of a system’s quality) shifts from 13.84 to 13.50 to a disappointing 9.47 in the future scenario.

The analysis of scoring systems is of interest to both the sportsperson and the mathematician. Indeed the branch of mathematics known as Game Theory is devoted to an analysis of the evolution and outcome of a multitude of different games, though the “games” are often economical or military in nature. If you are interested in some further reading you may wish to look at the book *Winning ways for your mathematical plays* by Berlekamp, Conway and Guy. The site [www.ics.uci.edu/~eppstein/cgt/](http://www.ics.uci.edu/~eppstein/cgt/) has many links to other games and their mathematical analysis. Another book full of fascinating problems is *Game Set and Math* by Ian Stewart where he tackles the tennis-scoring problem from the point of view of random walks.

The above model is unrealistic on several points. A player’s fitness will affect probabilities in long matches as will the tendency of some players to “choke” on big points. It is simply not true that a player’s probability of winning an individual point remains fixed during the course of a match.

Another issue (particularly in the men’s game) is that games tend to “go with service”. That is, a player’s probability of winning a point when serving is significantly higher than when receiving. The following exercise explores this feature.

**Exercise:** Suppose that you are a player in a Wimbledon final with a probability \( p_1 \) of winning a point when you serve and \( p_2 \) of winning a point when you receive. (Generally \( p_1 > p_2 \)). It follows from our above results that your probability \( g_1 \) of winning a

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service game is
\[ g_1 = p_1^4 + 4p_1^4(1 - p_1) + 10p_1^4(1 - p_1)^2 + 20p_1^5(1 - p_1)^3 \left\{ \frac{1}{1 - 2p_1(1 - p_1)} \right\} \]
and your probability of winning a game when receiving is similarly
\[ g_2 = p_2^4 + 4p_2^4(1 - p_2) + 10p_2^4(1 - p_2)^2 + 20p_2^5(1 - p_2)^3 \left\{ \frac{1}{1 - 2p_2(1 - p_2)} \right\} . \]

Let us now assume that you serve first and find the probability of winning an advantage set.

6-0: \[ g_1g_2g_1g_2g_1g_2 = g_1^3g_2^3 \]
6-1: You must win the last game and 5 of the first 6 games.

The 5 games won will be 3 on serve and 2 receiving or 2 on serve and 3 receiving. Thus the probability is
\[ 3C_3(g_1)^3(1 - g_1)^0 3C_2(g_2)^2(1 - g_2)g_1 + 3C_2(g_1)^2(1 - g_1) 3C_3(g_2)^3(1 - g_2)^0 g_1 \]
\[ = 3(g_1)^4(g_2)^2(1 - g_2) + 3(g_1)^3(g_2)^3(1 - g_1) \]
An interesting check on the result is to let \( g_1 = g_2 = g \). The probability becomes
\[ 3g^4g^2(1 - g) + 3g^3g^3(1 - g) = 6g^6(1 - g) \]
which correlates perfectly with our previous probability of a 6-1 win where the probability of winning a game (serving or receiving) was \( g \).

As an exercise try to generalise some of the other results to the situation where there are different probabilities for serving and receiving (What about the toss??). You can check your results by setting \( g_1 = g_2 = g \).

We have seen how a simple combination of binomial probability theory and standard computer software can be used to analyse different scoring systems in tennis. It is clear that an effective scoring system does much more than just keep the score! It also serves as a mechanism to favour the superior player by amplifying their edge. There are of course many other scoring systems in use in various games around the world. You may wish to apply a similar analysis to some other games and see how the table of \( w \) vs \( p \) shapes up for those systems. Once the mathematics is done it is fairly easy to use a spreadsheet to crank out the numbers. The derivative at \( p = 0.5 \) may be estimated by \( \frac{\Delta w}{\Delta p} \) for \( p = 0.5 \) and \( \Delta p \) small. As a challenge see if you can develop a different system for tennis with \( f'(0.5) > 13.84 \). If you have any questions or theories feel free to e-mail me at pahor@maths.unsw.edu.au.

References


Top Wimbledon Winning Countries. The United Kingdom tops the charts with the most champions at Wimbledon with 37 won games with the first championship title in 1877, and the last championship in 2016. The UK won 35 championships during the amateur era and two championships during the Open era. The United States is the second leading country by the number of Wimbledon Championships with 33 wins, its first title was won in 1920, and its last title in 2000. Watch the final game of Andy Murray's 2013 Wimbledon win over Novak Djokovic in full. More great content at http://www.wimbledon.com/index.html SUBSCRIBE to Winning at Wimbledon. As a result of temporary magical powers, you have made it to the Wimbledon finals and are playing Roger Federer for all the marbles. However, your powers cannot last the whole match. What score do you want it to be when they disappear, to maximize your chances of hanging on for a win? Show the answer. It sounds obvious that you should ask to be ahead two sets to love (it takes 3 out of 5 sets to win the men’s), and in the third set, ahead 5-0 in games and 40-love in the sixth game.